Some momentum polytopes for multiplicity free quasi-Hamiltonian manifolds

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Abstract

We discuss the classification of two families of multiplicity-free quasi-Hamiltonian manifolds, namely of rank one and the case where the moment map is surjective. In particular, this leads to a lot of new examples.

Preliminaries

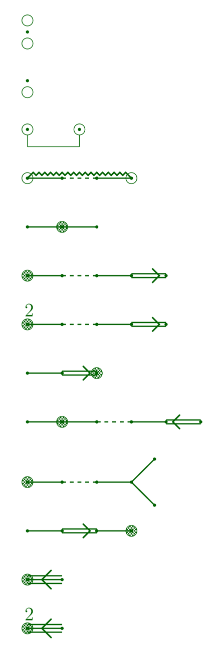
- K is simply connected compact Lie group with Lie algebra $\mathfrak{k}$ and complexification G
- $T_{\mathbb{R}}$ is a maximal torus for K with complexification T
- Let Λ be the weight lattice of K; this is the character group of T.
- A normal G-variety Z is called spherical if the Borel has a Z-open orbit.
- The **weight monoid** of Z can be identified with a submonoid of Λ.
- The **spherical roots** $\Sigma(Z)$ is the minimal set of primitive elements of $\Lambda(Z)$ such that

$$\mathcal{V}(Z) = \{\eta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda(Z), \mathbf{Q}) \mid \langle \eta, \sigma\rangle \leq 0 \forall \sigma \in \Sigma(Z)\}$$

where $\mathcal{V}(Z)$ denotes the **valuation cone** of Z.

Spherical roots

Spherical root	Diagram
α_1	
$2\alpha_1$	
$[1/2]\alpha + \alpha'$	
$\alpha_1 + \cdots + \alpha_r$	
$[1/2]\alpha_1 + 2\alpha_2 + \alpha_3$	
$\alpha_1 + \cdots + \alpha_r$	
$2\alpha_1 + \cdots + 2\alpha_r$	
$[1/2]\alpha_1 + 2\alpha_2 + 3\alpha_3$	
$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$	
$[1/2]2\alpha_1 + \cdots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$	
$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$	
$2\alpha_1 + \alpha_2$	
$4\alpha_1 + 2\alpha_2$	



Quasi-Hamiltonian manifolds

Definition. A quasi-Hamiltonian K-manifold is a smooth manifold M equipped with a K-action by twisted conjugation $x \mapsto kx\tau(k)^{-1}$, a 2-form w, and a smooth map $m : M \to K\tau$, called the (group valued)moment map with:

1. $d\omega = -m^*\eta$
2. $\iota(\xi_m)\omega = -\frac{1}{2}m^*(\theta^L + \theta^R)\xi, \xi \in \mathfrak{k}$
3. $\ker(\omega) \cap \ker \mathrm{d}\, m = \{0\}$

A quasi-Hamiltonian manifold is called multiplicity-free when all symplectic reductions are points.

Definition. Let K be a simply connected compact Lie group with automorphism τ and fundamental alcove $\mathcal{A}$. Let $\mathcal{P} \subseteq \mathcal{A}$ be a locally convex subset and Λ_S a lattice. Then $(\mathcal{P}, \Lambda_S)$ is called **spherical in** $a \in \mathcal{P}$ if

1. $\mathcal{P}$ is polyhedral in a, i.e.

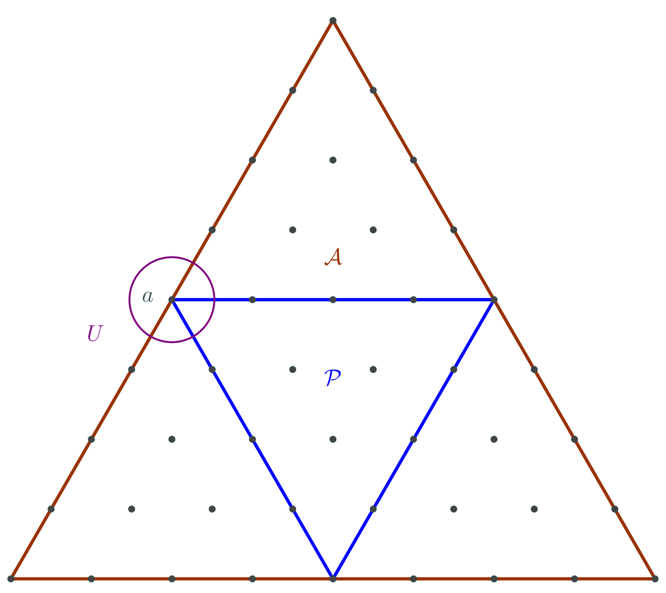
$$\mathcal{P} \cap U = (a + C_a\mathcal{P}) \cap U$$

for a neighborhood U of a in $\mathcal{A}$, and

2. there is a smooth affine spherical $K(a)_{\mathbb{C}}$-variety Z with weight monoid Γ_Z such that

$$C_a\mathcal{P} \cap \Lambda_S = \Gamma_Z$$

The pair $(\mathcal{P}, \Lambda_S)$ is called a **spherical pair** if it is spherical for all $a \in \mathcal{P}$.



Theorem (Knop 2016)**.** *There is a bijection between spherical pairs and multiplicity free quasi-Hamiltonian manifolds.*

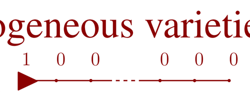
Momentum Polytopes of Rank one

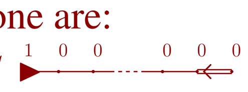
Local models

Lemma. *Let G, H be reductive. A smooth affine spherical G-variety Z of rank one is of exactly one of the following types:*

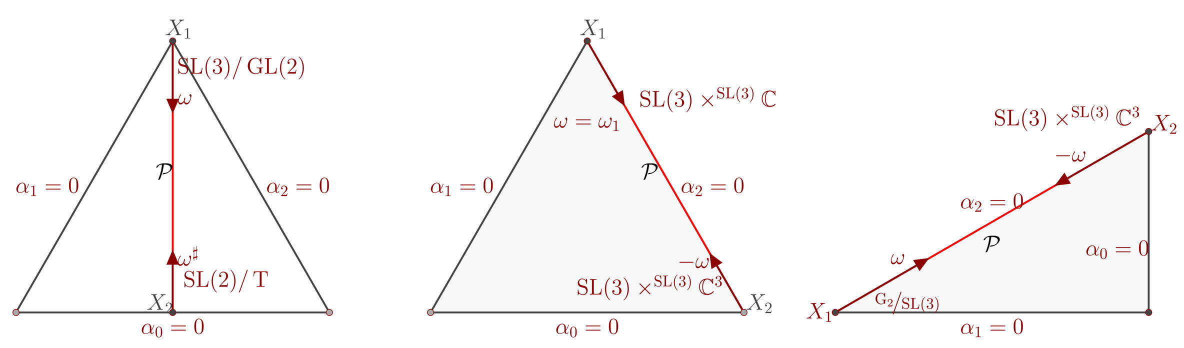
1. *$Z = G/H, V = 0$. We will call this the "homogeneous case", as it means that Z is a homogeneous spherical variety.*
2. *$Z = V = \mathbb{C}^n, H = G = \mathrm{SL}(n, \mathbb{C}), n \geq 1$ or $V = \mathbb{C}^{2n}, H = G = \mathrm{Sp}(2n, \mathbb{C}), n \geq 2$. We call this the "inhomogeneous case".*

- Homogeneous varieties of rank one correspond to the spherical roots.
- Inhomogeneous varieties of rank one are:

Type A 

Type C 

Examples

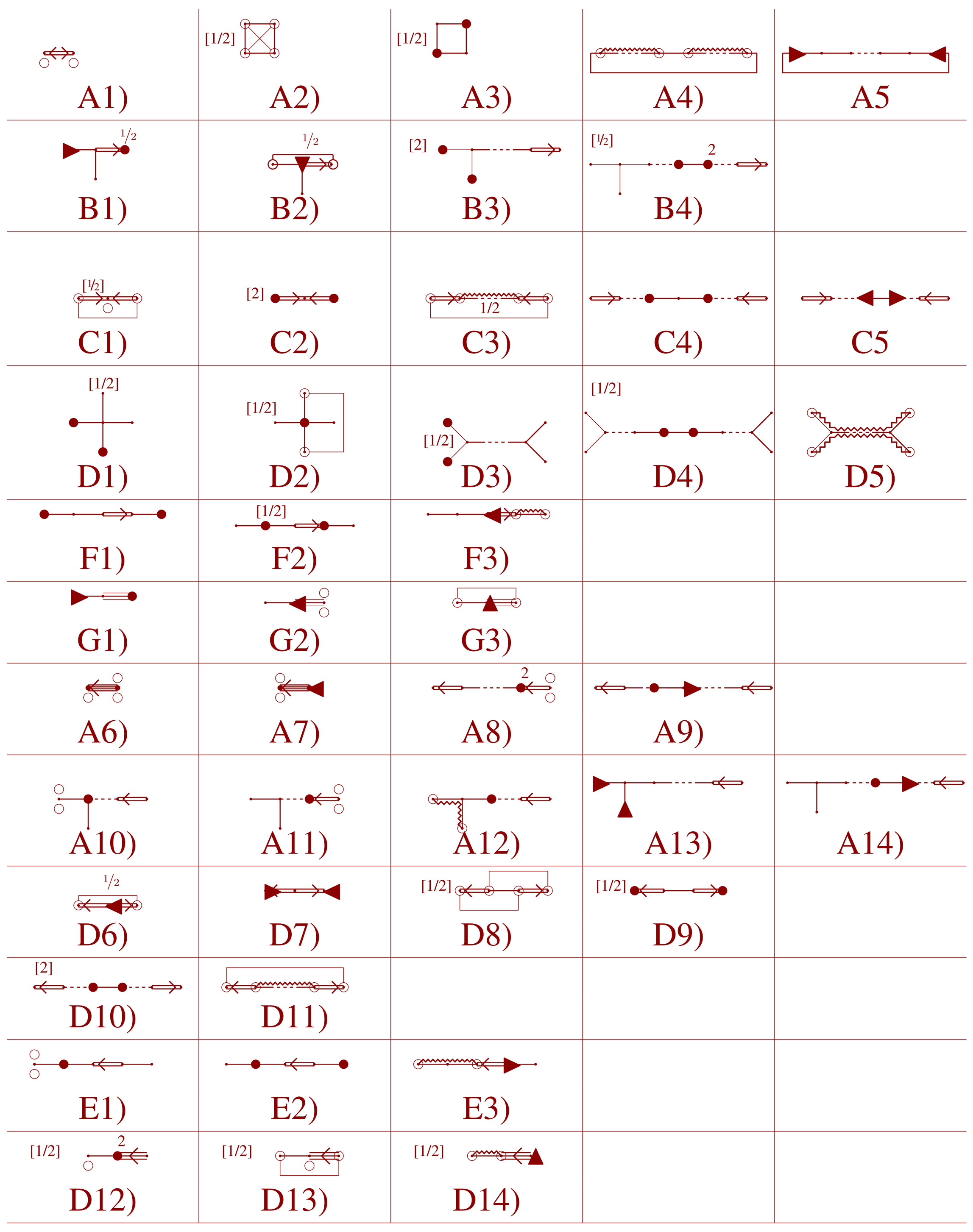


Classification

Theorem. *Necessary criterions for the existence of genuine quasi-Hamiltonians of rank one are:*

- *For all bi-homogeneous moment polytopes of rank one, $S(X_1)$ and $S(X_2)$ contain at least $n-1$ roots.*
- *Bi-inhomogeneous polytopes are only possible if $S(X_1) = S\backslash\{\alpha_j\}$ with $\langle\omega, \bar{\alpha}_k^\vee\rangle = 1$ and $S(X_2) := S\backslash\{\alpha_k\}$ with $\langle -\omega, \bar{\alpha}_j^\vee\rangle = 1$.*
- *Mixed momentum polytopes (homogeneous in X_1, inhomogeneous in X_2) can only exist if $|S(X_1)| = n$.*

Theorem. *The genuine quasi-Hamiltonian manifolds of rank one are (up to diagram automorphisms) exactly those in the table below:*



Surjective Moment Maps

- The moment map being surjective is equivalent to the fact that the moment polytope is the alcove $\mathcal{A}$.
- Local models are the smooth affine spherical G-varieties with G-saturated weight monoid and full rank.
- A submonoid Γ of Λ is called G-saturated if $\mathbb{Z}\Gamma \cap \Lambda^+ = \Gamma$.
- Pezzini and Van Steirteghem proved a combinatorial criterion to decide whether a given monoid is the weight monoid of a smooth affine spherical variety by only using standard invariants.
- We used this in a joint paper to classify all smooth affine spherical varieties of full rank with G-saturated weight monoid.
- This classification, the relations between simple roots and the criterion from above make the following characterization possible:

	Φ	Lattice
1A	$A_n^{(1)}$	$2\langle\bar{\alpha}_2, \bar{\alpha}_3, \ldots, \bar{\alpha}_n, d\omega_n\rangle_{\mathbb{Z}}$ with $d\vert n+1$
1B	$A_n^{(1)}$, n even	$\langle S^+ \oplus k\omega_{n-1}\rangle_{\mathbb{Z}}$ with $k\vert n+1$.
1C	n odd	$\langle\bar{\alpha}_2 + \bar{\alpha}_3, \ldots, \bar{\alpha}_{n-1} + \bar{\alpha}_n, e\omega_{n-1}, r\omega_{n-1} + \omega_n\rangle_{\mathbb{Z}}, e\vert\frac{n+1}{2}, 0 \leq r \leq e-1$.
2A	$B_n^{(1)}$	$2\mathbb{Z}S_0$
2B		$2\Lambda_0$
3A	$C_n^{(1)}$	$2\mathbb{Z}S_0$
3B		$2\Lambda_0$
3C		Λ_0
4A	$D_n^{(1)}$	$2\mathbb{Z}S_0$
4B		$2\Lambda_0$
4C		$2\langle\bar{\alpha}_1, \ldots, \bar{\alpha}_{n-2}, \bar{\alpha}_n, \omega_1\rangle_{\mathbb{Z}}$
4D	n even	$2\langle\bar{\alpha}_1, \ldots, \bar{\alpha}_{n-2}, \bar{\alpha}_n, \omega_n\rangle_{\mathbb{Z}}$
4E	n even	$2\langle\bar{\alpha}_1, \ldots, \bar{\alpha}_{n-2}, \bar{\alpha}_n, \omega_1 + \omega_n\rangle_{\mathbb{Z}}$
5A	$E_6^{(1)}$	$2\mathbb{Z}S_0$
5B		$2\Lambda_0$
6A	$E_7^{(1)}$	$2\mathbb{Z}S_0$
6B		$2\Lambda_0$
7	$E_8^{(1)}$	$2\Lambda_0$
8	$F_4^{(1)}$	$2\Lambda_0$
9	$G_2^{(1)}$	$2\Lambda_0$
10A	$A_4^{(2)}$	$\langle 2\omega_1, \omega_2\rangle_{\mathbb{Z}}$
10B	$A_{2n}^{(2)}$	$2\Lambda_0$
10C		Λ_0
11A	$A_{2n-1}^{(2)}$	$2\mathbb{Z}S_0$
11B	$A_{2n-1}^{(2)}$	$2\Lambda_0$
12A	$D_{n+1}^{(2)}$	$2\mathbb{Z}S_0$
12B		$2\Lambda_0$
12C		$\langle\omega_1, \ldots, \omega_{n-1}, 2\omega_n\rangle_{\mathbb{Z}}$
12D	n odd	$\langle\bar{\alpha}_1 + \bar{\alpha}_2, \bar{\alpha}_2 + \bar{\alpha}_3, \ldots, \bar{\alpha}_{n-1} + \bar{\alpha}_n, 2\bar{\alpha}_n\rangle_{\mathbb{Z}}$
13	$E_6^{(2)}$	$2\Lambda_0$
14	$D_4^{(3)}$	$2\Lambda_0$