Construction of multiplicity-free quasi-Hamiltonian manifolds

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Abstract

This talk describes how the combinatorial theory of spherical varieties can be used to construct new examples of (quasi-) Hamiltonian spaces. We start with an introducation to spherical varieties and their combinatorics. The second part of the talk will be about the classification of a family of smooth affine spherical varieties of rank one and the classification of so-called quasi-Hamiltonian model spaces, multiplicity free quasi-Hamiltonian manifolds with surjective moment map.

Introduction to spherical varieties

Definition

A normal G-variety is spherical if a Borel of G has a dense orbit on X.

Example

1. toric varieties $G = (\mathbb{C}^\times)^n$ and $x \in X$ with $G_x = \{e\}$. Here G is its own Borel.
2. $G = \mathrm{GL}(n)$ or $G = \mathrm{SL}(n)$ and $X = \mathbb{C}^n$ with standard linear action of G. Then $B_n \cdot e_n$ is dense in X.
3. $G = \mathrm{GL}(n), X = Symm(n \times n) := \{A \in Mat(n \times n) : A^t = A\}$ where G acts on X via $gA = (g^{-1})^t A g^{-1}$. Then $B_n \cdot id$ is dense in $Symm(n \times n)$.

Definition

A subgroup H of G is called spherical if G/H is a spherical G-variety. That means there is a Borel B of G such that BH is dense in G.

Example

Let $G = \mathrm{SL}(2)$ and $H = T$. Then H is the stabilizer of a point in $G/H = \mathbb{P}^1 \times \mathbb{P}^1 \setminus diag\mathbb{P}^1$, where G acts linearly and diagonally on each copy of $\mathbb{P}^1$.
The Borel B has an open orbit , namely the set of couples $(p, q \in \mathbb{P}^1 \times \mathbb{P}^1$ where $p \neq q$ and both are different from $[1, 0]$, the unique point fixed by B in $\mathbb{P}^1$.

Example

- $G \times G$ acting on $X = G$ by
$$(g_1, g_2)x = g_1 x g_2^{-1}$$
Using the Bruhat Decomposition, it follows: If B is a Borel for G and B^- is the opposite Borel, then $(B^- \times B)$ is a Borel for $G \times G$ and $(B^- \times B)e = B^- B$ is dense in $X = G$.
- Note that this is an example for an symmetric space:
If $\sigma : G \to G$ is an involution and $G^\sigma \subset H \subset N_G(G^\sigma)$, then G/H is called an symmetric space. In our example, we can take $\sigma : (g_1, g_2) \to (g_2, g_1)$ and get $(G \times G)^\sigma = diag(G)$ and $\frac{G \times G}{diag G} \cong G$.
- In general, every symmetric space is a spherical variety, and every symmetric subgroup is a sphericals subgroup.

Smooth Affine Spherical Varieties

- In this talk, we are particularly interested in **smooth affine spherical varieties**.
- Important subclasses have been known for a while:
  - all homogeneous varieties for G simple: Krämer 1979
  - simple spherical modules: Kac 1980
  - rank one: Akhiezer 1983
  - rank two: Wasserman 1996
- The classification was completed by Knop and Van Steirteghem (2006) in terms of spherical triples: By Luna's slice theorem, every smooth affine spherical variety is of the shape $G \times^H V$, and for every s.a.s. variety, $(\mathfrak{g}', \mathfrak{h}', V)$ is a spherical triple found in their lists.

Example

There are three smooth affine spherical varieties for $\mathrm{SL}(2)$, namely $\mathbb{C}^2, \mathrm{SL}(2)/T$ and $\mathrm{SL}(2)/N(T)$.

Invariants: Setting

Fix $T \subseteq B \subseteq G$.

1. $\Lambda = Hom(T, \mathbb{C}^\times) \equiv Hom(B, \mathbb{C}^\times)$, the weight lattice
2. The simple roots S

Invariants

- The lattice of X:

$$\Lambda(X) := \{\lambda \in \Lambda : \mathbb{C}(X)\text{contains a } B\text{-Eigenvector of weight}\lambda\}$$
$$:= \{\chi_f : f \in \mathbb{C}(X)^{(B)}\}.$$

  As X is spherical, for every $\lambda \in \Lambda(X)$, the space of B-eigenvectors in $\mathbb{C}(X)$ of weight λ is one-dimensional.
  The rank of the lattice is called the rank of X.

- The weight monoid of X.

$$\Gamma(X) := \{\lambda \in \Lambda^+ : \mathbb{C}[X]\text{contains a } B\text{-eigenvector of weight } \lambda.$$

  If X is quasi-affine, then $\mathbb{Z}\Gamma(X) = \Lambda(X)$.

Knop Conjecture, proved by Losev

A smooth affine spherical variety is uniquely determined by its weight monoid.

Example

Let $G = \mathrm{SL}(2)$ and $H = T$. The homogeneous space G/H is spherical and has rank 1. if we consider the B-stable open set:

$$\{([x,1][y,1]) : x \neq y\} \subset \mathbb{P}^1 \times \mathbb{P}^1 \setminus diag(\mathbb{P}^1) \cong G/H$$

then the function $\frac{1}{x-y}$ is in $\mathbb{C}(G/H)^{(B)}$ and its weight is α_1. So the lattice of G/H obviously is $Z\alpha_1$.

Example

Let $G = \mathrm{SL}(2)$ and $H = U$ the unipotent upper triangular matrices. Then

$$G/H \cong C^2 \setminus \{0, 0\}$$

where G acts lineary on $\mathbb{C}^2$. One can show that the function y, the second coordinate, is a B-eigenvector with weight ω_1, so $\Lambda(G/H) = \mathbb{Z}\omega_1$

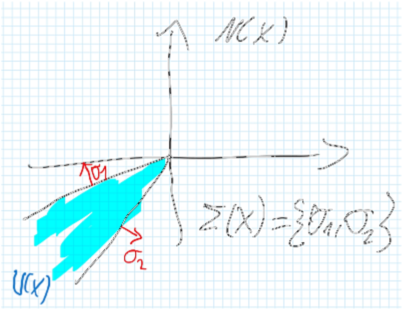
Spherical Roots

Spherical Roots

The set of **spherical roots** $\Sigma(Z)$ of Z is the minimal set of primitive elements of $\Lambda(Z)$ with

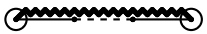
$$\mathcal{V}(Z) = \{\eta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda(Z), \mathbb{Q}) \mid \langle \eta, \sigma \rangle \leq 0 \forall \sigma \in \Sigma(Z)\}$$

where $\mathcal{V}(Z)$ is the so-called **valuation cone**.

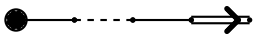
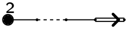
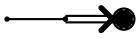
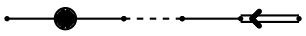
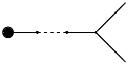


Spherical roots of smooth affine spherical varieties (1)

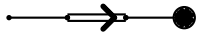
Spherical roots are exactly the weights of homogeneous smooth affine spherical varieties of rank one.

spherical root	Luna diagram
α_1	
$2\alpha_1$	
$[1/2]\alpha + \alpha'$	
$\alpha_1 + \cdots + \alpha_r$	
$[1/2]\alpha_1 + 2\alpha_2 + \alpha_3$	

Spherical roots of smooth affine spherical varieties (2)

spherical root	Luna diagram
$\alpha_1 + \cdots + \alpha_r$	
$2\alpha_1 + \cdots + 2\alpha_r$	
$[1/2]\alpha_1 + 2\alpha_2 + 3\alpha_3$	
$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$	
$[1/2]2\alpha_1 + \cdots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$	

Spherical roots of smooth affine spherical varieties (3)

spherical root	Luna diagram
$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$	
$2\alpha_1 + \alpha_2$	
$4\alpha_1 + 2\alpha_2$	

Introduction to quasi-Hamiltonian Manifolds

Notation

- K is a semisimple and simply connected Lie-group with Lie-algebra $\mathfrak{k}$ and complexification G
- $T_{\mathbb{R}}$ is a maximal torus for K with complexification T
- Let Λ be the weight lattice of K; it is a the character group of T,that can be identified with the character group of the Borel B.
- The set of simple roots of G (for a fixed torus) is called S.

Quasi-Hamiltonian manifolds

Definition

A **(twisted) quasi-Hamiltonian** $K\tau$**-manifold** (this means that K acts on itself by twisted conjugation) is a smooth manifold M equipped with a K-action, a 2-form w, and a smooth map $m : M \to K$, called the (group valued) **moment map** such that

1. m is K-equivariant
2. w is K-invariant and satisfies $dw = -m^*\eta$
3. $\iota(\xi_m)w = -\frac{1}{2}m^*(\theta^L + \theta^R)\xi, \xi \in \mathfrak{k}$
4. $\ker(w) \cap \ker \mathrm{d}\, m = \{0\}$

where θ^L, θ^R are the left- resp. right-invariant Maurer-Cartan-form and η is the canonical Cartan-3-form.

In this talk, *all quasi-Hamiltonian manifolds are assumed to be multiplicity free* which means that all symplectic reductions are points.

Why spherical varieties in this talk?

Theorem

Smooth affine spherical varieties are local models for multiplicity free quasi-Hamiltonian manifolds. (Knop 2016)

Invariants of quasi-Hamiltonian Manifolds

A multiplicity free quasi-Hamiltonian manifold is uniquely determined by two invariants:

- the image of the invariant moment map, the **moment polytope** $\mathcal{P}$ and
- the **generic isotropy group** L

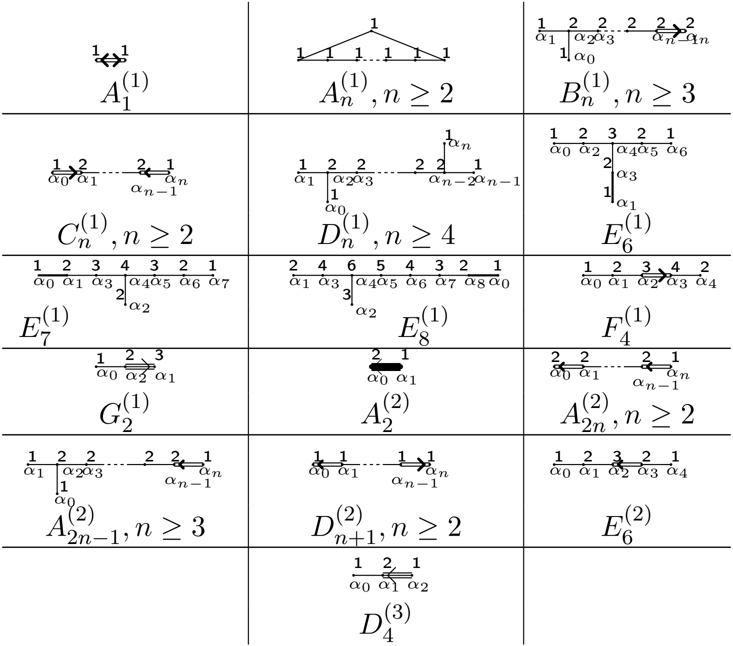
[Knop 2016].

The moment map

For a moment map $m : M \to K$ we introduce the invariant moment map $m_+ : M \to \bar{\mathcal{A}}$ which makes the following diagramm commute:

$$
\begin{array}{ccc}
M & \xrightarrow{\;m\;} & K \\
{\scriptstyle m_+}\downarrow & & \downarrow \\
\bar{\mathcal{A}} & \xrightarrow[\text{bij}]{} & K/_{\tau}K
\end{array}
$$

Here, $\bar{\mathcal{A}}$ is the alcove of an affine root system uniquely determined by K and $\operatorname{ord}\tau$. It is well known that this alcove is in bijection with the orbits of the twisted action.


$A_1^{(1)}$
$A_n^{(1)}, n \geq 2$
$B_n^{(1)}, n \geq 3$
$C_n^{(1)}, n \geq 2$
$D_n^{(1)}, n \geq 4$
$E_6^{(1)}$
$E_7^{(1)}$
$E_8^{(1)}$
$F_4^{(1)}$
$G_2^{(1)}$
$A_2^{(2)}$
$A_{2n}^{(2)}, n \geq 2$
$A_{2n-1}^{(2)}, n \geq 3$
$D_{n+1}^{(2)}, n \geq 2$
$E_6^{(2)}$
$D_4^{(3)}$

The generic isotropy group

Fact:

The generic isotropy L_S can be encoded in a sublattice Λ_S of the weight lattice Λ of K.

Spherical Pairs

We call $(\mathcal{P}, \Lambda_M)$ **spherical in** $a \in \mathcal{P}$ if:

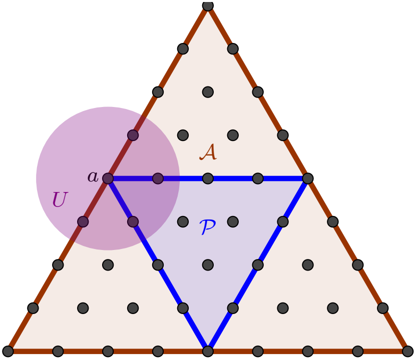
- $\mathcal{P}$ is polyhedral in a, that means

  $$\mathcal{P} \cap U = (a + C_a\mathcal{P}) \cap U$$

  in a neighborhood U of $a \in \mathcal{A}$.
- There is a smooth affine spherical $K(a)_{\mathbb{C}}$-variety Z with weight monoid Γ_Z and

  $$C_a\mathcal{P} \cap \Lambda_M = \Gamma_Z$$

The pair $(\mathcal{P}, \Lambda_M)$ is called **a spherical pair** if it is spherical for every $a \in \mathcal{P}$.



Knop 2016

The map $M \to (\mathcal{P}, \Lambda_M)$ gives a bijection between multiplicity free quasi-Hamiltonian manifolds and spherical pairs.

Moment Polytopes of Rank one

The Setting

We look for triplets (X_1, X_2, ω) with:

- The momentum polytope, the line segment $\mathcal{P} = [X_1 X_2]$, touches every wall of the alcove. We call these manifolds **genuine**.
- The monoid $\mathbb{N}\omega$ is the weight monoid of a smooth affine spherical $K^{\mathbb{C}}_{X_1}$-variety and $-\mathbb{N}\omega$ is the weight monoid of smooth affine spherical $K^{\mathbb{C}}_{X_2}$-variety.

Local Models of Rank one

Lemma

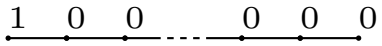
A smooth affine spherical variety $G \times^H V$ of rank one is of one of the following types

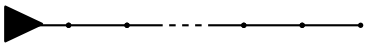
- $Z = G/H, V = 0$, so **homogeneous** variety, their weights are the spherical roots.
- $Z = V = \mathbb{C}^n, H = G = \mathrm{SL}(n, \mathbb{C})$ or $V = \mathbb{C}^{2n}, H = G = \mathrm{Sp}(2n, \mathbb{C})$, we call these **inhomogeneous** varieties.

The inhomogeneous varieties

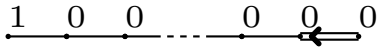
The weights of the inhomogeneous smooth affine spherical varieties of rank one are, using the classification by Kac(1980):

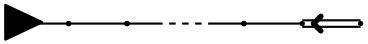
- Type A





- Type C:



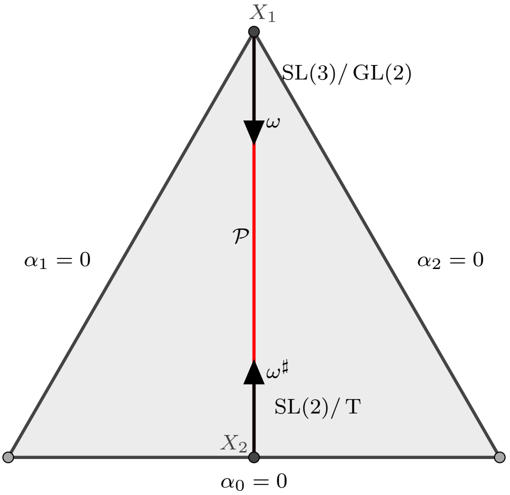


Example

- Let $\Phi = A_2^{(1)}$, $X_1 = (0,0,0)$, $S(X_1) = \{\overline{\alpha}_1, \overline{\alpha}_2\}$.
- $\omega = \overline{\alpha}_1 + \overline{\alpha}_2 = (1, 0, -1)$ corresponds to ${}^{\mathrm{SL}(3)}/_{\mathrm{GL}(2)}$.
- $X_2 = (1, 0, -1)$ with local root system $\Phi \setminus \{\alpha_0\}$.
- $\omega^\sharp = -\overline{\alpha}_1 - \overline{\alpha}_2 = \overline{\alpha}_0$ corresponds to ${}^{\mathrm{SL}(2)}/_{T}$

We have found a bi-homogeneous manifold.

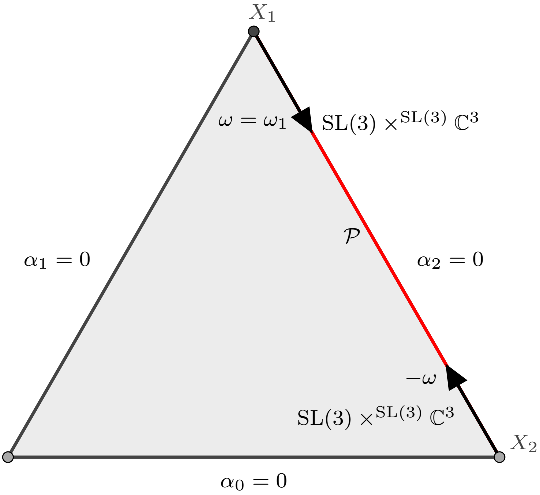
Bi-homogeneous Polytope for $A_2^{(1)}$



Example

- $\Phi = A_n^{(1)}$, $X_1 = (0,0,0,\dots,0)$, $\Phi(X_1) = S \setminus \{\alpha_0\}$.
- Choose $\omega = \omega_1$ corresponding to $\mathrm{SL}(n+1) \times^{\mathrm{SL}(n+1)} \mathbb{C}^{n+1}$
- $\Phi(X_2) = S \setminus \{\alpha_1\}$
- There, $-\omega$ is a local model, again corresponding to $\mathrm{SL}(n+1) \times^{\mathrm{SL}(n+1)} \mathbb{C}^{n+1}$

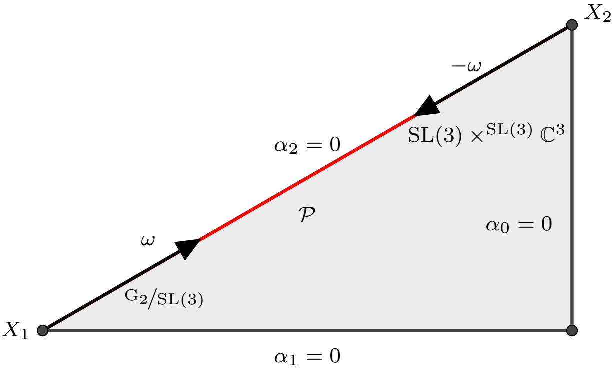
Bi-inhomogeneous Polytope for $A_2^{(1)}$



Example

- We consider $\Phi = \mathrm{G}_2^{(1)}$, $X_1 = (0,0,0)$
- $\omega = \overline{\alpha}_2 + 2\overline{\alpha}_1$ is a homogeneous model, the corresponding variety is ${}^{\mathrm{G}_2}\!/_{\mathrm{SL}(3)}$.
- Hence $S(X_2) = \{\alpha_0, \alpha_2\}$, and $-\omega$ is an inhomogeneous model corresponding to $\mathrm{SL}(3) \times^{\mathrm{SL}(3)} \mathbb{C}^3$ there.

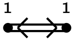
Mixed Polytope for $G_2^{(1)}$



Structure Lemmas [K.P.]

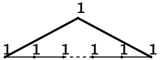
- For a bi-homogeneous moment polytope, $S(X_1)$ and $S(X_2)$ contain at least $n-1$ simple roots.
- Bi-inhomogeneous polytopes have $S(X_1) = S \setminus \{\alpha_j\}$ with $\langle \omega, \overline{\alpha}_k^\vee \rangle = 1$ and $S(X_2) := S \setminus \{\alpha_k\}$ with $\langle -\omega, \overline{\alpha}_j^\vee \rangle = 1$.
- Mixed polytopes (homogeneous in X_1, inhomogeneous in X_2) fulfill $|S(X_1)| = n$.

Example: $A_1^{(1)}$



$S(X_1)$	ω	$S(X_2)$	$\omega^\sharp$ **hom.?**	$-w$ **inhom.?**
α_1	I_1	α_0		$-\omega = I_0 \Rightarrow$**A5**
α_1	$\overline{\alpha}_1$	α_0	$\omega^\sharp = \alpha_0 \Rightarrow$**A4**	
α_1	$2\overline{\alpha}_1$	α_0	$\omega^\sharp = 2\overline{\alpha}_0 \Rightarrow$**A1**	

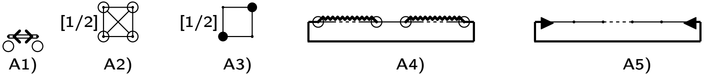
Example: $A_n^{(1)}$



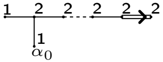
$S(X_1)$	ω	$S(X_2)$	$\omega^\sharp$ **hom.?**	$-w$ **inhom.?**
$S \setminus \{\alpha_0\}$	$\overline{\alpha}_{1,n}$	$S\setminus\{\alpha_1, \alpha_n\}$	$\omega^\sharp = \overline{\alpha}_0 \Rightarrow$ **A4**	
	I_1	$S \setminus \{\alpha_1\}$		$-\omega = I_0 \Rightarrow$ **A5**
$S \setminus \{\alpha_0\}$, $n = 3$	$[\frac{1}{2}](\overline{\alpha}_1+2\overline{\alpha}_2+\overline{\alpha}_3)$	$S \setminus \{\alpha_2\}$	$\omega^\sharp = [\frac{1}{2}]\overline{\alpha}_3 + 2\overline{\alpha}_0 + \overline{\alpha}_1 \Rightarrow$ **A3**	
$S\setminus\{\overline{\alpha}_1, \overline{\alpha}_n\}$	$2\overline{\alpha}_0$	$S \setminus \{\alpha_0\}$	**no**: $\omega^\sharp = 2\alpha_{1,n}$	**no**: $\langle -\omega, \overline{\alpha}_1^\vee \rangle = \langle -\omega, \overline{\alpha}_n^\vee \rangle > 0$
$S \setminus \{\alpha_d, \alpha_e\}$; $d \le e$	$\overline{\alpha}_{d+1,e-1}$	$S \setminus \{\alpha_{d+1}, \alpha_{e-1}\}$	$\omega^\sharp = \alpha_{e,d} \Rightarrow$ **A4**	
$S \setminus \{\alpha_0, \alpha_4\}$, $n \ge 4$	$[\frac{1}{2}]\overline{\alpha}_1 + 2\overline{\alpha}_2 + \overline{\alpha}_3$	$S \setminus \{\alpha_2\}$	**no**: $\omega^\sharp = [\frac{1}{2}]2\overline{\alpha}_0+\overline{\alpha}_1+\overline{\alpha}_3+2\overline{\alpha}_4 + \cdots + 2\overline{\alpha}_n$	**no**: $\langle -\omega, \alpha_3^\vee \rangle = 0$, $\langle -\omega, \overline{\alpha}_4^\vee \rangle = [\frac{1}{2}]1$
$S \setminus \{\alpha_1, \alpha_3\}, n = 3$	$[\frac{1}{2}](\overline{\alpha}_1 + \overline{\alpha}_3)$	α_2, α_0	$\omega^\sharp = [\frac{1}{2}](\overline{\alpha}_2 + \overline{\alpha}_0) \Rightarrow$ **A2**	
$S \setminus \{\alpha_1, \alpha_3\}, n > 3$	$[\frac{1}{2}](\overline{\alpha}_1 + \overline{\alpha}_3)$	$S\setminus\{\alpha_1, \alpha_3\}$	**no**: $\omega^\sharp = [\frac{1}{2}](\overline{\alpha}_0 + \overline{\alpha}_{2,n})$	**no**: $\langle -\omega, \overline{\alpha}_2^\vee \rangle > 0, \langle -\omega, \overline{\alpha}_3^\vee \rangle > 0$

Quasi-Hamiltonians for $A_n^{(1)}$

	ω	$\omega^\sharp$
A1)	$2\overline{\alpha}_0 \in H(1)$ $X_1 = (\frac{1}{2}, -\frac{1}{2})$	$2\overline{\alpha}_1 \in H(0)$ $X_1 = (0, 0)$
A2)	$[\frac{1}{2}](\overline{\alpha}_1 + \overline{\alpha}_3) \in H(0,2)$ $X_1 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4})$	$[\frac{1}{2}](\overline{\alpha}_0 + \overline{\alpha}_2) \in H(1,3)$ $X_2 = (1/2, 0, 0 - 1/2)$
A3)	$[2]\frac{1}{2}(\overline{\alpha}_1 + 2\overline{\alpha}_2 + \overline{\alpha}_3) \in H(0)$ $X_1 = (0,0,0,0)$	$[2]\frac{1}{2}(\overline{\alpha}_3 + 2\overline{\alpha}_0 + \overline{\alpha}_1) \in H(2)$ $X_2 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$
A4)	$\overline{\alpha}_1 + \cdots + \overline{\alpha}_{d-1} \in H(0,d)$ $X_1 = (a, \ldots, \underset{d}{a}, b, \ldots, b)$ $a = \frac{n+1-d}{2n+2}, b = \frac{-d}{2n+2}$	$\overline{\alpha}_d + \cdots + \overline{\alpha}_n + \overline{\alpha}_0$ $\in H(d-1, 1)$ $X_2 = (a, b, \ldots, \underset{d-1}{b}, c, \ldots, c)$ $a = \frac{2n+2-d}{2n+2},\ b = \frac{n+1-d}{2n+2},\ c = \frac{-d}{2n+2}$
A5)	$\boldsymbol{\omega = I_0(1)}$ $X_1 = (\frac{n}{n+1}, -\frac{1}{n+1}, \ldots, -\frac{1}{n+1})$	$\boldsymbol{-\omega = I_1(0)}$ $X_2 = (0, \ldots, 0)$



Example: $B_n^{(1)}$



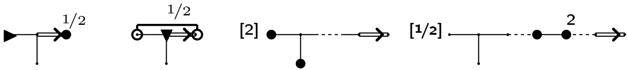
$S(X_1)$	ω	$S(X_2)$	$\omega^\sharp$ hom.?	$-w$ inhom.?
$S \setminus \{\alpha_0\}$	$[2]\overline{\alpha}_{1,n}$	$S \setminus \{\alpha_1\}$	$\omega^\sharp = [2]\overline{\alpha}_0 + \overline{\alpha}_{2,n}$ $\Rightarrow$B3	
$n = 3$	$\frac{1}{2}(\overline{\alpha}_1 + 2\overline{\alpha}_2 + 3\overline{\alpha}_3)$	$S \setminus \{\alpha_3\}$		$-\omega = I_0 \Rightarrow$B1
$n = 3$	$\overline{\alpha}_1 + 2\overline{\alpha}_2 + 3\overline{\alpha}_3$	$S \setminus \{\alpha_3\}$	no: $\omega^\sharp = \frac{3}{2}\overline{\alpha}_0 + \frac{1}{2}\overline{\alpha}_1 + \overline{\alpha}_2$	no: $\langle -\omega, \overline{\alpha}_0^\vee \rangle = 2$
$S \setminus \{\alpha_0, \alpha_1\}$	$[2]\overline{\alpha}_{2,n}$	$S \setminus \{\alpha_2\}$	$\omega^\sharp = [2]\frac{1}{2}(\overline{\alpha}_0 + \overline{\alpha}_1) \Rightarrow$B4	
$S \setminus \{\alpha_0, \alpha_1\}$ für $n = 4$	$[\frac{1}{2}]\overline{\alpha}_2 + 2\overline{\alpha}_3 + 3\overline{\alpha}_4$	$S \setminus \{\alpha_n\}$	no: $\omega^\sharp = [\frac{1}{2}](\frac{3}{2}\overline{\alpha}_0 + \frac{3}{2}\overline{\alpha}_1 + 2\overline{\alpha}_2 + \overline{\alpha}_3$	no: Type D
$S \setminus \{\alpha_0, \alpha_2\}$	$\omega \in C(1)$	never contains α_0 and α_2		

$n=3$	$[\frac{1}{2}]\overline{\alpha}_1+\overline{\alpha}_3$	$S\backslash\{\alpha_1,\alpha_3\}$	no: $\omega^\sharp=[\frac{1}{2}]\overline{\alpha}_0+2\overline{\alpha}_2+\overline{\alpha}_3$	no: $\langle-\omega,\overline{\alpha}_2^\vee\rangle=2$
$S\setminus\{\alpha_0,\alpha_d\}, 2<d<n$	$\omega\in C(2)$	$S\setminus\{\alpha_1,\alpha_{d-1}\}$	no: $\omega^\sharp=2\overline{\alpha}_0+\overline{\alpha}_{2,d-1}+2\overline{\alpha}_{d,n}$	
	$\omega\in C(n)$	$\alpha_0\notin S(X_2)$		
$S\backslash\{\alpha_0,\alpha_n\}$	$\omega\in C(2)$	Never contains α_0 and α_n		
$S\setminus\{\alpha_{d+1}\}, 1\leq d+1\leq n-1$	$[\frac{1}{2}]\overline{\alpha}_0+\overline{\alpha}_1+2\overline{\alpha}_{2,d}$	$S\setminus\{\alpha_d\}$	$\omega^\sharp=[\frac{1}{2}]2\overline{\alpha}_{d+1,n}$ $\Rightarrow$B4	
$d=n-3$	$[\frac{1}{2}]\overline{\alpha}_{n-2}+2\overline{\alpha}_{n-1}+3\overline{\alpha}_n$	$S\setminus\{\alpha_n\}$	no: $\omega^\sharp=[\frac{1}{2}]2\overline{\alpha}_0+2\overline{\alpha}_1+4\overline{\alpha}_{2,n-3}+3\overline{\alpha}_{n-2}+2\overline{\alpha}_{n_1}+\overline{\alpha}_n$	no: Typ D
$S\setminus\{\alpha_2\}, n=3$	$\frac{1}{2}\overline{\alpha}_1+\overline{\alpha}_3$	$S\backslash\{\alpha_1,\alpha_3\}$		$-\omega=I_2$ $\Rightarrow$B2
	$\overline{\alpha}_1+\overline{\alpha}_3$	$S\backslash\{\alpha_1,\alpha_3\}$	no: $\omega^\sharp=\overline{\alpha}_0+2\overline{\alpha}_2+\overline{\alpha}_3$	no:$\langle-\omega,\overline{\alpha}_2^\vee\rangle=2$
$d=2$	$[2]\overline{\alpha}_0$	$S\setminus\{\alpha_0\}$	no: $\omega^\sharp=\overline{\alpha}_1+2\overline{\alpha}_{2,n}$	no: $\langle-\omega,\overline{\alpha}_2^\vee\rangle>0$,
$S\setminus\{\alpha_d,\alpha_{d+1}\}$	$\omega\in C(0)$	$\alpha_{d+1}\notin S(X_2)$		
	$\omega\in C(n)$	$\overline{\alpha}_d\notin S(X_2)$		

$S\setminus\{\alpha_d, \alpha_e\}$ $e \neq d+1$	$\omega \in C(0)$	$\alpha_e \notin S(X_2)$		
	$\omega \in C(d+1)$		No	No
	$\omega \in C(n)$	$\alpha_d \notin S(X_2)$		
$S\setminus\{a_d, \alpha_n\}$	$\omega \in C(0)$ **oder** $C(0)+C(1)$	$\alpha_n \notin S(X_1)$		
	$\omega \in C(d+1)$		No	No
$S \setminus \{\alpha_n\}$	$[\frac{1}{2}]\overline{\alpha}_0 + \overline{\alpha}_1 + 2\overline{\alpha}_{2,n-1}$	$S \setminus \{a_{n-1}\}$	$\omega^\sharp = [2]\overline{\alpha}_n \Rightarrow$B4	
$n=3$	$\overline{\alpha}_{0,2}$	$S\setminus\{\overline{\alpha}_0, \overline{\alpha}_2\}$	no: $\omega^\sharp = \overline{\alpha}_2+2\overline{\alpha}_3$	no: $\langle -\omega, \overline{\alpha}_3^\vee \rangle = 2$

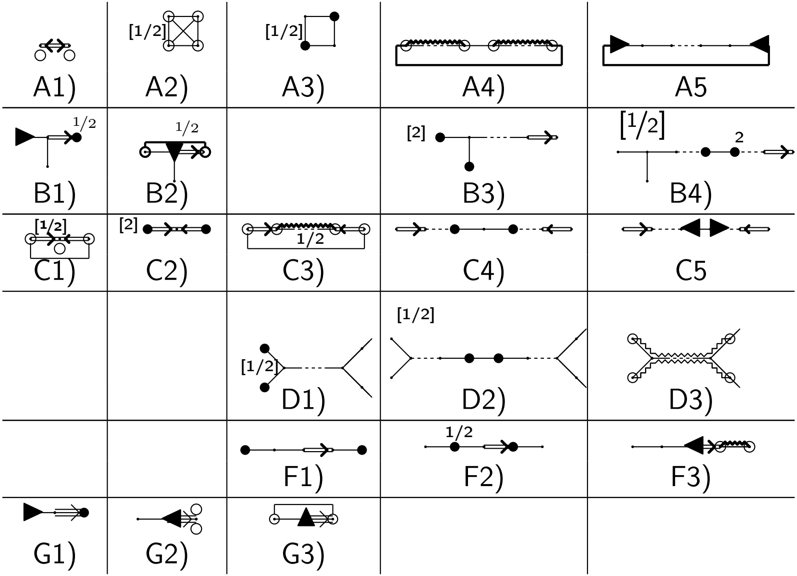
Moment polytopes for $B_n^{(1)}$

	ω	$\omega^\sharp$
B1)	$\frac{1}{2}(\overline{\alpha}_0 + 2\overline{\alpha}_2 + 3\overline{\alpha}_3) \in H(1)$ $X_1 = (1, 1, 0)$	$\boldsymbol{-\omega \in I_1(3)}$ $X_2 = (1/2, 1/2, 1/2)$
B2)	$\frac{1}{2}(\overline{\alpha}_1 + \overline{\alpha}_3) \in H(2)$ $X_1 = (1/2.1/2, 0)$	$\boldsymbol{-\omega \in I_2(1, 3)}$ $X_2 = (3/4, 1/4, 1/4)$
B3)	$[2](\overline{\alpha}_1 + \cdots + \overline{\alpha}_n) \in H(0)$ $X_1 = (0, \dots, 0)$	$[2](\overline{\alpha}_0 + \overline{\alpha}_2 + \cdots + \overline{\alpha}_n) \in H(1)$ $X_2 = (1, 0, \dots, 0)$
B4)	$[2](\frac{1}{2}(\overline{\alpha}_0 + \overline{\alpha}_1) + \overline{\alpha}_2$ $+ \cdots + \overline{\alpha}_d) \in H(d+1)$ $X_1 = (\underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{d+1}, 0, \dots, 0)$	$[2](\overline{\alpha}_{d+1} + \cdots + \overline{\alpha}_n) \in H(d)$ $X_2 = (\underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{d}, 0, \dots, 0)$



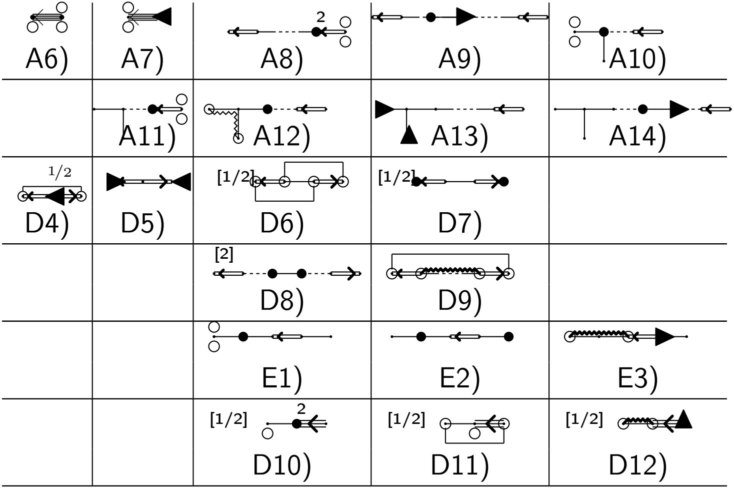
B1) B2) B3) B4)

The List



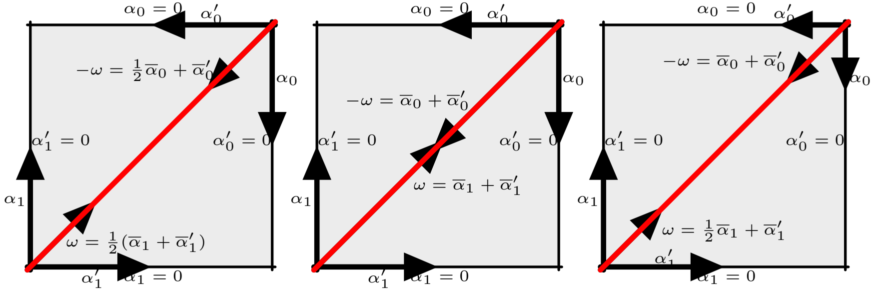
A1)
[1/2] A2)
[1/2] A3)
A4)
A5
1/2 B1)
1/2 B2)
[2] B3)
[1/2] 2 B4)
[1/2] C1)
[2] C2)
1/2 C3)
C4)
C5
[1/2] D1)
[1/2] D2)
D3)
F1)
1/2 F2)
F3)
G1)
G2)
G3)

The List 2


A6)
A7)
2
A8)
A9)
A10)
A11)
A12)
A13)
A14)
1/2
D4)
D5)
[1/2]
D6)
[1/2]
D7)
[2]
D8)
D9)
E1)
E2)
E3)
[1/2]
2
D10)
[1/2]
D11)
[1/2]
D12)

The semisimple case

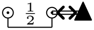
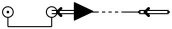
Let Φ be a product of affine root systems with two or more connected components. Then all genuine multiplicity free quasi-Hamiltonian manifolds of rank one are:



$A_1^{(1)} \times A_1^{(1)}$ $\quad$ $A_1^{(1)} \times A_1^{(1)}$ $\quad$ $A_2^{(2)} \times A_2^{(2)}$

All local models are $(\mathfrak{sl}(2) \times \mathfrak{sl}(2), \Delta\mathfrak{sl}(2), 0)$.

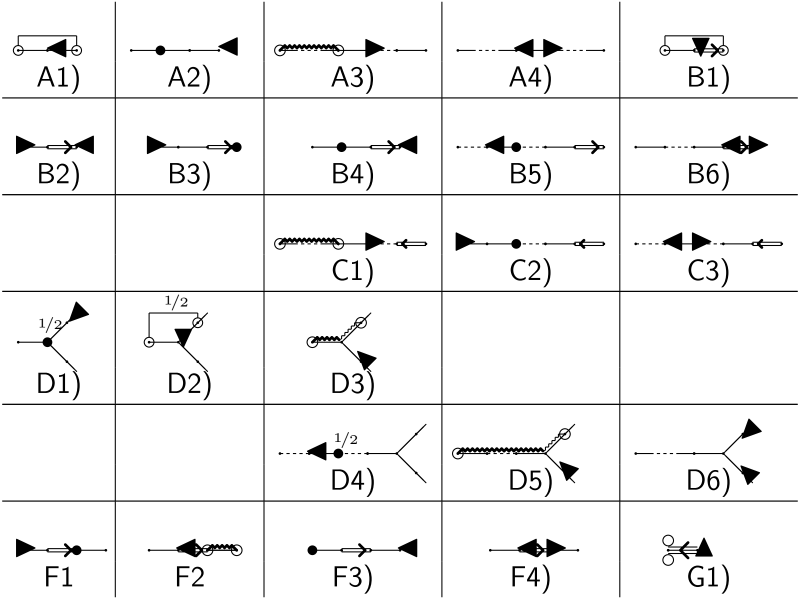
All spherical pairs of rank one where the root system is a product of affine and classical root systems with at least one connected component of each type and $\mathcal{P}$ hits every wall of the alcove are those corresponding to:

	diagram	type	$-\omega$	model in X_2
(1)		$A_1 \times A_1^{(1)}$	$\in I_1(0)$	$(\mathfrak{sl}(2), \mathfrak{sl}(2), \mathbb{C}^2)$
(2)		$A_1 \times C_n^{(1)}$	$\in I_1(0)$	$(\mathfrak{sp}(2n), \mathfrak{sp}(2n), \mathbb{C}^{2n})$
(3)		$A_1 \times G_2^{(1)}$	$\in I_2(0)$	$(\mathfrak{sl}(3), \mathfrak{sl}(3), \mathbb{C}^3)$
(4)		$A_1 \times A_2^{(2)}$	$\in I_1(0)$	$(\mathfrak{sl}(2), \mathfrak{sl}(2), \mathbb{C}^2)$
(5)		$A_1 \times A_{2n}^{(2)}$	$-\omega \in I_1(0)$	$(\mathfrak{sp}(2n), \mathfrak{sp}(2n), \mathbb{C}^{2n})$

Hamiltonian manifolds of rank one

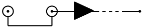
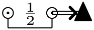
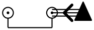
- Every Hamiltonian manifold can be seen as a quasi-Hamiltonian by using the exponential map as a moment map.
- Therefore, we can use our theory to classify Hamiltonian manifolds of rank one in a very similar way.
- There can not be any bi-homogeneous manifolds, as simple roots are linearly independent.
- The counterparts of inhomogeneous local models have a local root system with $n-1$ simple roots.

Hamiltonian Manifolds of rank one



Hamiltonian-semisimple

All Hamiltonian manifolds of rank one for K semisimple, not simple, where $\mathcal{P}$ hits every wall of the dominant chamber are:

	diagram	type	$-\omega$	model in X_2
(1)		$A_1 \times A_n$	$\in I_2(1)$	$(\mathfrak{sl}(n), \mathfrak{sl}(n), \mathbb{C}^n)$
(2)		$A_1 \times C_n$	$\in I_2(1)$	$(\mathfrak{sp}(2n), \mathfrak{sp}(2n), \mathbb{C}^{2n})$
(3)		$A_1 \times B_2$	$\in I_2(1)$	$(\mathfrak{sl}(2), \mathfrak{sl}(2), \mathbb{C}^2)$
(4)		$A_1 \times G_2$	$\in I_2(1)$	$(\mathfrak{sl}(2), \mathfrak{sl}(2), \mathbb{C}^2)$

For X_1 corresponding to $\omega = \alpha + \alpha_1'$, the local model is always $(\mathfrak{sl}(2) \times \mathfrak{sl}(2), \Delta\mathfrak{sl}(2), 0)$.

Quasi-Hamiltonain Model Spaces

Quasi-Hamiltonian Model Space

Recall:

A (quasi-affine) G-variety for G reductive is called a **model variety** if its coordinate ring contains every irreducible representation exactly once. These varieties have played a crucial role for a broad range of classification problems.

Definition

A quasi-Hamiltonian $K\tau$-manifold is called a **(quasi-Hamiltonian) model $K\tau$-space** if it is multiplicity free and its moment map is surjective.

What we want

Let Λ_M be a sublattice of Λ with $\operatorname{rk}\Lambda_M = \operatorname{rk}\Lambda$. Let $\mathcal{C}_a\mathcal{A}$ denote the tangent cone of $\mathcal{A}$ in $a \in \mathcal{A}$ and $K(a)_{\mathbb{C}}$ the complexification of the K-stabilizer of a. The pair $(\mathcal{A}, \Lambda_M)$ is in bijection with a quasi-Hamiltonian manifold if and only if for every vertex a of $\mathcal{A}$, the monoid $\mathcal{C}_a\mathcal{A} \cap \Lambda_M$ is the weight monoid of a smooth affine spherical $K(a)_{\mathbb{C}}$-variety Z.

How we do it

1. We determine all possible local models in one vertex of the alcove $\mathcal{A}$, we choose $(0, \ldots, 0)$. This gives candidates for the lattices Λ_M, these are a subclass of smooth affine spherical varieties of full rank. This was done in a joint paper with G. Pezzini and B. Van Steirteghem.
2. We decide which of these lattices correspond to G-saturated smooth affine spherical varieties of full rank in every other vertex.

The local models

1. We say that a submonoid Γ of Λ^+ is G-**saturated** if
$$\mathbb{Z}\Gamma \cap \Lambda^+ = \Gamma$$
Observe that, under this assumption, the lattice $\mathbb{Z}\Gamma$ also determines Γ.
2. We call a monoid $\Gamma \subseteq \Lambda^+$ **smooth** if it is the weight monoid of a smooth affine spherical variety, that is there exists a unique smooth affine G-variety Z such that $\mathbb{C}[Z] \cong \oplus_{\lambda\in\Gamma} V(\lambda)$ as a G-module.
3. We say that a sublattice $\mathcal{X}$ of Λ has **full rank** if $\operatorname{rk}\mathcal{X} = \operatorname{rk}\Lambda$. Furthermore, we say that a submonoid of Λ^+ has *full rank* if the lattice it generates in Λ has full rank. Finally, we say that an affine spherical G-variety Z has *full rank* if its weight monoid has full rank.

For the rest of this talk, we assume all weight monoids to be G-saturated and to have full rank.

The tool we will use for our classification is a combinatorial criterion, introduced by Pezzini and Van Steirteghem based on work by Camus, to decide whether a given monoid of dominant weights can be realized as the weight monoid of a smooth affine spherical variety.

We will need a special set of spherical roots for the smoothness criterion, the so-called N-spherical roots $\Sigma^N(\Gamma)$ associated to Γ. By definition, it is equal to $\Sigma(G/N_G(H))$ where G/H is the open orbit of the most generic affine spherical variety with weight monoid Γ.

Lemma:

Let Γ be a submonoid of Λ^+: If $\operatorname{rk}\mathbb{Z}\Gamma = \operatorname{rk}\Lambda$ and Γ is G-saturated, then $\Sigma^N(\Gamma) \subset 2S \cup S^+$ with

$$2S := \{2\overline{\alpha} : \alpha \in S\}$$

$$S^+ := \{\overline{\alpha} + \beta : \alpha, \beta \in S, \alpha \neq \beta, \overline{\alpha} \not\perp \beta\}$$

Lemma

Let $\bar{\sigma} \in 2S \cup S^+$, let Γ be a G-saturated submonoid of Λ^+ of full rank. Then $\bar{\sigma} \in \Sigma^N(\Gamma)$ if and only if:

1. $\bar{\sigma} \in \mathbb{Z}\Gamma$; and
2. if $\bar{\sigma} = 2\overline{\alpha}$ then $\langle \overline{\alpha}^\vee, \gamma \rangle \in 2\mathbb{Z}$ for all $\gamma \in \mathbb{Z}\Gamma$;

Lemma

Let $\Gamma \subseteq \Lambda^+$ be a G-saturated submonoid. Among all the subsets $F \subseteq S$ such that the relative interior of the cone spanned by $\{\overline{\alpha}^\vee|_{\mathbb{Z}\Gamma} : \overline{\alpha} \in F\}$ in $\mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Q})$ intersects the valuation cone $\mathcal{V}(\Gamma)$, there is a unique one, denoted S_Γ, which contains all the others.

The Smoothness Criterion

Theorem

Let Γ be a G-saturated monoid of dominant weights of G, let Γ have full rank and let $\Sigma^N(\Gamma), S_\Gamma$ as defined above. Then Γ is the weight monoid of a smooth affine spherical G-variety if and only if

1. $\{\overline{\alpha}^\vee|_{\mathbb{Z}\Gamma} : \overline{\alpha} \in S_\Gamma\}$ *is a* **subset of a basis** *of* $\mathbb{Z}\Gamma^*$,
2. *if* α, β *in* S_Γ *and* $\alpha \neq \beta$, *then* $\alpha \perp \beta$, *and*
3. *if* $\alpha \in S_\Gamma$, *then* $2\alpha \notin \Sigma^N(\Gamma)$

This is Pezzini's and Van Steirteghem's criterion adapted to our setting.

Example

We investigate the family of monoids for G of type C_4 and $\Sigma^N(\Gamma) = S^+ = \{\overline{\alpha}_1 + \overline{\alpha}_2, \overline{\alpha}_2 + \overline{\alpha}_3, \overline{\alpha}_3 + \overline{\alpha}_4\} := \{\sigma_1, \sigma_2, \sigma_3\}$. We want to show that Γ is smooth if and only if $\Gamma = \Lambda^+$

First, we determine S_Γ:

	$\overline{\alpha}_1 + \overline{\alpha}_2$	$\overline{\alpha}_2 + \overline{\alpha}_3$	$\overline{\alpha}_3 + \overline{\alpha}_4$
$\overline{\alpha}_1^\vee$	1	-1	0
$\overline{\alpha}_2^\vee$	1	1	-1
$\overline{\alpha}_3^\vee$	-1	1	0
$\overline{\alpha}_4^\vee$	0	-1	1

So, for sure $\overline{\alpha}_1, \overline{\alpha}_3 \in S_\Gamma$, and $\overline{\alpha}_2 \notin S_\Gamma$ because $\sum_i \langle \overline{\alpha}_2^\vee, \sigma_i \rangle > 0$. If $\overline{\alpha}_4$ were in S_Γ, so were $\overline{\alpha}_2$, but this is impossible. So $S_\Gamma = \{\overline{\alpha}_1, \overline{\alpha}_3\}$.
In particular, conditions 2 and 3 of the criterion are fulfilled.

	$\overline{\alpha}_1 + \overline{\alpha}_2$	$\overline{\alpha}_2 + \overline{\alpha}_3$	$\overline{\alpha}_3 + \overline{\alpha}_4$
$\overline{\alpha}_1^\vee$	1	-1	0
$\overline{\alpha}_2^\vee$	1	1	-1
$\overline{\alpha}_3^\vee$	-1	1	0
$\overline{\alpha}_4^\vee$	0	-1	1

Adding up all spherical roots shows $\omega_2 \in \Gamma$. Then, by looking at $\overline{\alpha}_3 + \overline{\alpha}_4$, also $\omega_4 \in \Gamma$. But then, condition (1) of the smoothness criterion yields $\Gamma = \Lambda^+$.

Theorem by PPVS

The G-saturated smooth weight monoids of full rank for G simple and simply connected correspond to the lattices $\mathbb{Z}\Gamma$ with:

1. $2\mathbb{Z}S \subseteq \mathbb{Z}\Gamma \subseteq 2\Lambda$ for every G
2. $\mathbb{Z}\Gamma = \langle S^+ \oplus \mathbb{Z}(k\omega_{n-1})\rangle$ for A_n, n even
3. $\mathbb{Z}\Gamma = \langle \overline{\alpha}_2 + \overline{\alpha}_3, \overline{\alpha}_3 + \overline{\alpha}_4, \ldots, \overline{\alpha}_{n-1} + \overline{\alpha}_n, e\omega_{n-1}, r\omega_{n-1} + \omega_n\rangle_{\mathbb{Z}}$ for A_n, n odd
4. $\langle \overline{\alpha}_1 + \overline{\alpha}_2, \ldots, \overline{\alpha}_{n-1} + \overline{\alpha}_n, 2\overline{\alpha}_n\rangle$ and $\langle \omega_1, \ldots, \omega_{n-1}, 2\omega_n\rangle$ for B_n
5. $\mathbb{Z}\Gamma = \Lambda$ for C_n

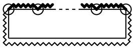
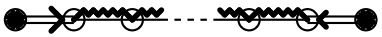
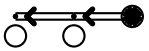
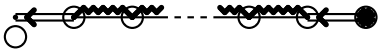
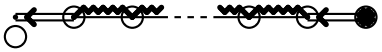
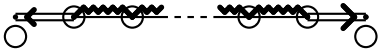
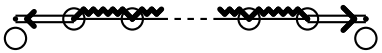
How to classify:

- The theorem gives a list of the monoids of local models in X_0. They can be divided in 24 cases.
- For every case, we decide if the lattice generated by this monoid corresponds to a spherical pair in every other vertex of the alcove.
- Using the relations in the affine root system, we can show:
  - We have $\Lambda_0 \subseteq \Lambda_k \ \forall k = 1, \dots, n$ (and hence, also $2\Lambda_0 \subseteq 2\Lambda_k$) for every connected affine root system. In other words, the lattice spanned by fundamental weights in X_0 is a sublattice of the lattice spanned by fundamental weights in every other vertex.
  - If Φ is of type $A_n^{(1)}$ and $C_n^{(1)}$, we have $\Lambda_0 = \Lambda_k \forall k = 1, \dots, n$.

  This observation settles almost all cases (not $A_{2n}^{(2)}$) with the lattices (1)
- The other cases are case by case studies using the smoothness criterion.

Theorem

Let K be simple and simply connected and τ be a smooth automorphism of K. Let $\Lambda = \langle \omega_1, \ldots, \omega_n \rangle$ be its weight lattice and S its set of simple roots. Then the map $M \to \Lambda_M$ gives a bijection between quasi-Hamiltonian model $K\tau$-spaces and the lattices in the following table:

	K	$\mathrm{ord}(\tau)$	Φ	Λ_M	Luna Diagram
1	any combination but $K = \mathrm{SU}(2n+1)$, $\mathrm{ord}(\tau) = 2$		any but $A_{2n}^{(2)}$	$2\mathbb{Z}S \subseteq \Lambda_M \subseteq 2\Lambda$	all $2S$
2	$\mathrm{SU}(n+1)$, n even	1	$A_n^{(1)}$	$\Lambda_M = \langle \overline{\alpha}_1 + \overline{\alpha}_2, \overline{\alpha}_2 + \overline{\alpha}_3, \ldots, \overline{\alpha}_{n-1} + \overline{\alpha}_n, k\omega_{n-1} \rangle$ with $k \| n+1, k \in \mathbb{Z}_{\geq 1}$	
3	$\mathrm{SU}(n+1)$, n odd	1	$A_n^{(1)}$	$\Lambda_M = \langle \overline{\alpha}_2 + \overline{\alpha}_3, \overline{\alpha}_3 + \overline{\alpha}_4, \ldots, \overline{\alpha}_{n-1} + \overline{\alpha}_n, e\omega_{n-1}, r\omega_{n-1} + \omega_n \rangle$ with $r, e \in \mathbb{Z}_{\geq 1}, e \| \frac{n+1}{2}$, $0 \leq r \leq e-1$	
4	$\mathrm{Sp}(2n)$	1	$C_n^{(1)}$	$\Lambda_M = \Lambda$	
5	$\mathrm{SU}(5)$	2	$A_4^{(2)}$	$\Lambda_M = \langle 2\omega_1, \omega_2 \rangle$	
6	$\mathrm{SU}(2n+1)$	2	$A_{2n}^{(2)}$	$\Lambda_M = \Lambda$	
7	$\mathrm{SU}(2n+1)$	2	$A_{2n}^{(2)}$	$\Lambda_M = 2\Lambda$	
8	$\mathrm{Spin}(2n+2)$	2	$D_{n+1}^{(2)}$	$\Lambda_M = \langle \omega_1, \omega_2, \ldots, \omega_{n-1}, 2\omega_n \rangle$	
9	$\mathrm{Spin}(2n+2)$ n odd	2	$D_{n+1}^{(2)}$	$\Lambda_M = \langle \overline{\alpha}_1 + \overline{\alpha}_2, \overline{\alpha}_2 + \overline{\alpha}_3, \ldots, \overline{\alpha}_{n-1} + \overline{\alpha}_n, 2\overline{\alpha}_n \rangle$	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